Handy Formulas

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Introduction

Most of the time, opticians work with their hands and eyes. You may have chosen this profession because you like it that way, and prefer it to working with a calculator or computer. Nevertheless, optical components are specified in terms of geometry and units. They behave according to physical laws. You'll occasionally need to make calculations.

In many shops the formulas for these calculations are on scraps of paper taped to the walls. Are they correct? Under what conditions do they work? Who said so? When? And what happens when the walls are painted?

In this module we'll present some of the most relevant formulas, and how and when to use them. Some of these will be familiar to most people in the shop but some will be new because they're either difficult to find or haven't been published elsewhere.

Sagitta

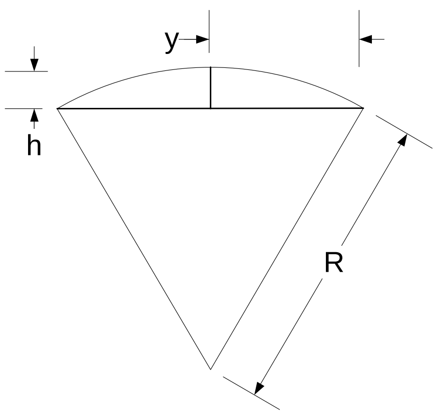

Figure 1: Sagitta

All spheres

Eq. 1 $h = (y^2/R) / [1 + \sqrt{(1 - y^2/R^2)}]$

Where h = sagitta,
R = radius of curvature
And y = ½ of the span

This is accurate for all spheres.[1]

When using a spherometer with ball feet

Eq. 2 $h = (y^2/\mathcal{R}) / [1 + \sqrt{(1 - y^2/ \mathcal{R}^2)}]$

Where
h = sagitta,
$\mathcal{R}$ = R-B

R = radius of curvature: *Positive for concave radii, negative for convex radii*
B = radius (half the diameter) of the contact ball
y = half-span of the ball circle at the zenith of the balls. (When the balls touch a plane surface, their points of contact lie on a circle whose diameter is 2y.)

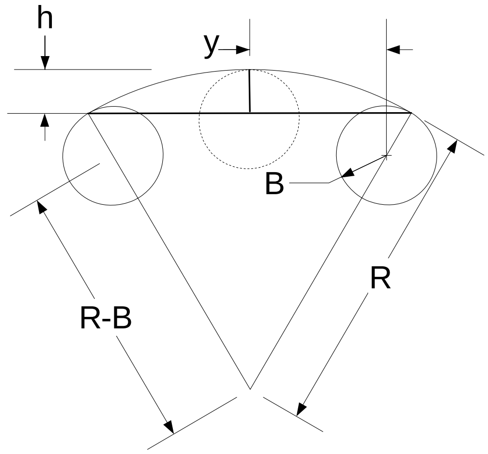

Figure 2: Sagitta with ball-foot spherometer

Handy approximation for R>>ϕ

Eq. 3 $h \cong \phi^2/8R$
Eq. 4 $R \cong \phi^2/8h$

Where h = sagitta
ϕ = diameter of part (or full span of segment being measured)
R = radius of curvature

This approximation is very useful when R is substantially larger than ϕ, because it's so simple you don't even need a calculator. When R = ϕ, it's only off 7%. When R = 5ϕ, it's off by 0.25%, and it gets much better fast.

Slope at edge of lens or block

Eq. 5 $\theta = \cos^{-1}(|h/R| - 1)$
Eq. 6 $\theta = \sin^{-1}(y/R)$

Where θ = angle between surface normals at center and edge of lens or block
h = sagitta
y = half-diameter of lens or block
R = radius of curvature
| | means absolute value: Use a plus sign for what's between the bars

Test plate fit

Eq. 7 $\Delta R = 4N\lambda R^2/\phi^2$
Eq. 8 $N = \phi^2\Delta R/4R^2\lambda$

Where ΔR = difference in radius between part and test plate
N = number of fringes of power showing
ϕ = diameter of interfering area

And R = nominal radius of curvature.

Use the same units throughout! If your radius is expressed in millimeters, then use millimeters for the diameter and the wavelength also. HeNe red = 0.0006328 mm = 6.328×10^{-4} mm.

Bevels

Cup wheel radius for small bevel on circular part

The most common way to put a bevel on a circular part is to grind the part's OD against a cup wheel. Bevels produced in this way aren't conical as drawn on the print, but replicate the spherical shape of the cup. This is close enough for small bevels. For larger bevels see the next section.

Eq. 9 $$R = \phi/(2\cos\theta)$$

Where ϕ = part diameter
R = radius or curvature of cup wheel
And θ = the angle between the bevel and the barrel edge, at the barrel edge.

For 45° bevels,

Eq. 10 $$R = \phi/\sqrt{2}$$

Cup wheel radius for large 45° bevel on circular part

Eq. 11 $$R = \sqrt{(\phi/2 - \phi b + b^2)}$$

Where b = the leg length of the bevel measured from the edge of the part directly toward the center of the part. This will produce the desired angle at the midline of the bevel.[2]

Cup wheel radius for 45° bevel on the cut chord of a D-shape spherical part

Sometimes part of a circular lens is cut off to form a D-shape. If you want to put an accurate, even, 45° bevel on the cut section, use a cup with this radius and hold the part against the cup so that it fits. (It's easier than it sounds.) This can be done for convex or concave radii.[3]

Eq. 12 $$R_C = \sqrt{[2(R_L^2 - y^2)]}$$

Where R_C = radius of curvature of cup wheel
R_L = radius of curvature of lens on side being beveled
And y = lateral offset of cut from barrel axis.

Tilt of concave radius part rim-mounted against a decentered bevel

When a bevel is wider on one side than the other, the bevel rim is lower on the wider side. When blocking such a part against a plate, or rim-mounting it in a lens barrel, this situation causes the lens to tilt. Surprisingly there's a very simple relationship that doesn't depend on the overall width of the bevel or its angle:

Eq. 13 $$\alpha = (b_2 - b_1)/2R$$

Where b_2 = maximum width of the bevel measured laterally from part edge towards part center
b_1 = minimum width of the bevel measured laterally from part edge towards part center
And R = radius of curvature of the surface with the bevel.
α is in units of radians. To convert to degrees, take the $\sin^{-1}$ of α.

This result is helpful when deciding whether rim mounts or wax blocking will be sufficient to meet tilt specifications.[4]

Blocking

Capacity of a plano block for circular parts

Start by laying parts across a diameter of the block, with proper spacing between. Count the number of parts across the diameter.

Eq. 14 $$N_B = (N_D)^2/1.3$$

Where N_B = number of parts that will fit on the block, properly spaced
And N_D = number of parts that fit across a diameter of the block with same spacing between.[5]

Capacity of a radius block for circular parts

Eq. 15 $$N_B \cong 5.2\ R^2(1-\cos\alpha)/\phi^2$$

Where N_B = approximate number of parts that will fit on the block, properly spaced
R = radius of curvature of *part* if concave; or of *blocking tool* if convex
ϕ = diameter of part
And α = slope angle of block at outer edge, recommended not to exceed 60° unless unavoidable.

Of course, you won't get 5.2 parts in any block. Choose the closest number. The capacity of blocks with small numbers of parts may not be exact; adjust angles and spacing as necessary.

Slope angle at edge of spherical blocks containing 3 or 4 pieces

Eq. 16 $$\alpha_3 = \sin^{-1}[(\phi+a)/(R*\sqrt{3})] + \sin^{-1}(\phi/2R)$$
Eq. 17 $$\alpha_4 = \sin^{-1}[(\phi+a)/(R*\sqrt{2})] + \sin^{-1}(\phi/2R)$$

Where α_3 = slope angle at outer edge of 3-lens block
α_4 = slope angle at outer edge of 4-lens block
ϕ = the diameter of the parts
a = the separation between lenses
R = radius of parts

Wedge in spherical parts off-center of a block due to grinding removal[6]

Eq. 18 $$\alpha = \sin^{-1}(\Delta h*D_C/2R^2)$$

Where α = angle shift in a given row of parts
Δh = vertical height removal
D_C = distance from center to center of parts in row under consideration, across diameter
R = part radius of curvature

Change of thickness required to adjust for wedge in spot tool

Sometimes a spot tool produces wedged parts. What is the ideal thickness to minimize wedge in a given tool? Rearranging equation 18, we get:

Eq. 19 $$\Delta h = 2R^2\alpha/D_C$$
Eq. 20 $$\Delta h = 2R^2W/\phi D_C$$

Where W = the edge thickness variation in parts within the row under consideration
ϕ = part diameter.

Fringe scaling factors

Here's where we separate the opticians from the poseurs. This subject is broadly – almost universally – misunderstood. The first thing to understand is this:

> **One fringe represents one wave of optical path difference between the interfering wavefronts, at the plane of interference. Always.**

It's true. Always. So why do people say "a fringe is a half wave"? They're not even wrong! They're just not completing the thought. It should be this:

> A fringe, as seen in a test plate or Fizeau interferometer or other system in which the light goes out and returns along the same path, represents one wave of optical path difference at the plane of interference. One half of that optical path difference was incurred on the way out, and one half on the way back. So that fringe is equivalent to $\lambda/2$ of surface height variation in reflection, or $\lambda/2$ of transmitted wavefront deviation per pass.

But who has time to say that?

There are other situations in which one fringe – which *always* equals one wave of accumulated optical path difference – is caused by different amounts of surface distortion.[7]

In the following situations, H is the height of the perturbation. Put H in terms of λ, and the formulas will give results in terms of λ = fringes. The fringe scaling factor in the interferometer is the ratio of the specified parameter – either H or TWF or RWF, to the number of waves of accumulated optical path difference (OPD) or fringes. For example, when flatness is specified, the tolerance is on the height deviation. When measured in double-pass in first surface reflection, the ratio is OPD = 2H1 (see Equation 21 below.) Therefore the fringe scaling factor is 0.5.

Normal-incidence optical path lengths

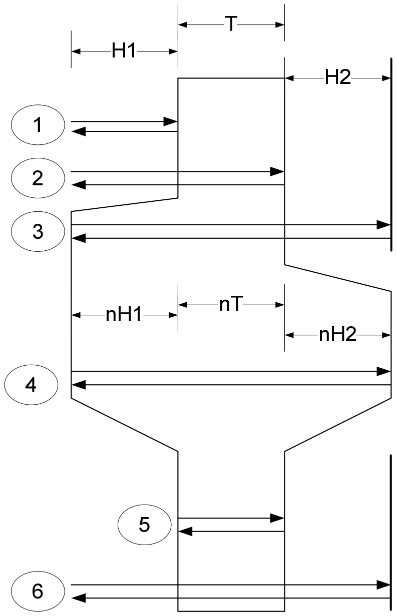

Figure 3: Various light paths to consider in normal-incidence optical path difference calculation

Eq. 21	(1 vs. reference plane)	OPD first surface reflection = 2H1
Eq. 22	(2 vs. 4)	OPD 2^{nd} surface reflection = 2(n - 1)H1 + 2nH2
Eq. 23	(4 vs. 5)	Internal fringes = 2n(H1 + H2)
Eq. 24	(3 vs. 6)	OPD double pass transmission = 2(n-1)H1
Eq. 25	(4 vs. 6)	OPD double-pass transmission = 2(n - 1)(H1 + H2)

Note: These formulas assume that the exterior index of refraction = 1. If that is not true – for example, if the part is immersed in water or bonded with optical cement – then use the versions below:

Eq. 26	(1 vs. reference plane)	OPD first surface reflection = $2n_1H1$
Eq. 27	(2 vs. 4)	OPD 2^{nd} surface reflection = $2(n_2 – n_1)H1 + 2n_2H2$
Eq. 28	(4 vs. 5)	Internal fringes = $2n_2(H1 + H2)$
Eq. 29	(3 vs. 6)	OPD double pass trans. = $2[(n_2 - n_1)H1]$
Eq. 30	(4 vs. 6)	OPD double-pass trans. = $2[(n_2 - n_1)H1 + (n_3 – n_2)H2]$

Where n_1 = index to the left of the enclosed object
n_2 = index within the enclosed object
and n_3 = index to the right of the enclosed object.

Ratio of internal reflection OPD to double-pass transmission OPD

Here's an interesting and handy result I discovered. You can determine the transmitted wavefront from viewing internal reflection fringes by using this simple ratio:

Eq. 31 $\quad OPD_{double\text{-}pass\ transmission} / OPD_{internal} = (n-1)/n$

Or, when immersed in another index,

Eq. 32 $\quad OPD_{double\text{-}pass\ transmission} / OPD_{internal} = (n_{internal} – n_{external}) / n_{internal}$

Transmitted wavefront distortion due to index inhomogeneity

Why waste all your time making perfect surfaces when the glass is "lumpy" inside? What homogeneity grade does the glass need to be to achieve a given transmitted wavefront?

Eq, 33 $OPD_{material}$ per pass = $t*\delta n/\lambda$

Where t = thickness
δn = index inhomogeneity
λ = wavelength

Oblique-incidence optical path lengths

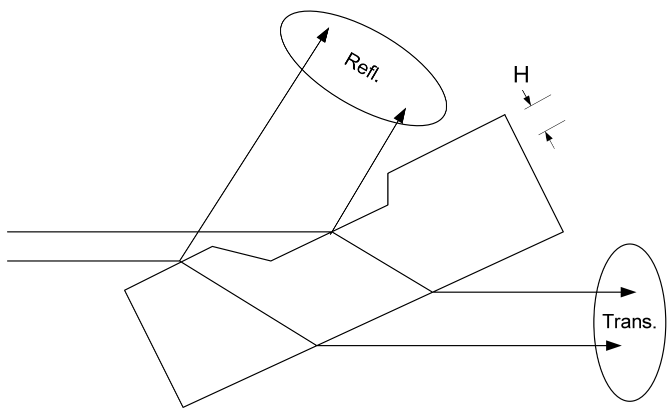

Figure 4: Oblique light paths in reflection and transmission

Eq. 34 Oblique reflected wavefront *per pass*, OPD = $2H\cos\alpha$
Eq. 35 Oblique transmitted wavefront *per pass*[8], OPD = $H[\sqrt{(n^2 - \sin^2\alpha)} - \cos\alpha]$

Where α = external angle of incidence.

Note: These formulas assume that the exterior index of refraction = 1. If that is not true – for example, if the part is immersed in water or bonded with optical cement – then replace 1 in the equations with n(external) and n with n(internal).

Note: The defect is drawn on the entry face, but Equation 29 works the same in either direction. If the part is not parallel, use the *external* angle of incidence appropriate for each surface.

Mechanical distortions of optical components

Distortion due to thermal gradient

When one side of a window or other flat parallel component is at a different temperature from the other side, the component will bend due to thermal expansion[9]. If the component is unconstrained,

Eq. 36 $h = y^2\alpha\Delta T/2t$

Where h = sagittal height of distortion
y = half-diameter of part, or half-diameter of aperture of interest
α = coefficient of thermal expansion
ΔT = difference of temperatures
t = thickness of part

Equation 36 can be rearranged as

Eq. 37 $h = (y/2t)*y\alpha\Delta T$

This form shows more directly how the bending of the part is affected by its aspect ratio. The disc is bent into a spherical section whose radius is

Eq. 38 $R = t/\alpha\Delta T$

Where R = resultant radius of curvature of distorted component.

Distortion due to pressure or self-weight

Windows can be called upon to resist pressure; common examples being a porthole, sight glass, airplane window, or laser end window. The amount that the window bends depends upon the way it's mounted, the pressure differential, the material properties, and the window's dimensions. The window may be "clamped," meaning that it is rigidly embedded around its diameter (for example, epoxied or fritted around its edge,) or it may be "simply supported," meaning that it is supported by a ring around its edge (for example, an O-ring.) These two cases produce very different amounts of bending.

The equations below are substantial simplifications of the full expressions[10]; nevertheless they give nearly identical results[11].

Eq. 39 $h_{ss} \cong 5/4\ Py^4/(Et^3)$
Eq. 40 $h_c \cong 0.176\ Py^4/(Et^3)$

Where h_{ss} = sag height of simply supported window
H_c = sag height of clamped window
P = pressure
y = half-diameter of circular window
E = Young's modulus
t = thickness

Make sure to use the same units for Young's modulus as for pressure.

The force of pressure acts the same as the force due to the component's own weight if it is mounted with its face horizontal. For self-weight sag, replace P with ρt, where ρ is the specific gravity of the window's material.

Pressure-bearing window thickness and "safety factors"

Put a window under too much pressure and it will break. Thicker windows resist more pressure. The minimum thickness to resist a given pressure is given below:

Eq. 41 $t_c/\phi = 0.866\sqrt{(SF*P/4Y)}$

Eq. 42 $t_{ss}/\phi = 1.06\sqrt{(*SF*P/4Y)}$

Where t_{ss} = thickness, simply-supported window
t_c = thickness, clamped window
ϕ = diameter of unsupported region
SF = "safety factor"
P = pressure
Y = window material yield strength.

Note that yield strength is *not* the same as Young's modulus. It can be orders of magnitude lower.

The term "safety factor," although in common use, is misleading because it doesn't guarantee a particular margin of safety. More accurately, what is called the safety factor is a *fudge factor* that we *hope* is large enough to make up for what we don't know. The above analysis is theoretical and based on perfect situations. Scratches, cracks, material imperfections, imperfect support, thermal stresses, and fatigue all reduce the strength of the window. "Safety factor" should never be set lower than 4; and higher if failure would result in injury, mission failure, or great expense.

Beam deviations by optical components

Beam displacement by inclined plate

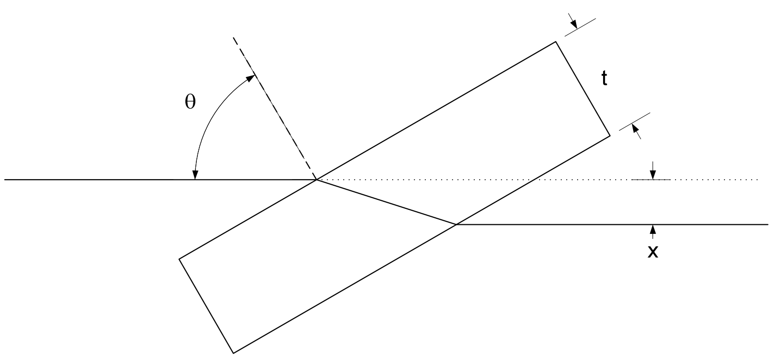

Figure 5: Beam displacement by inclined plate

Eq. 43 $x = t \sin\theta[1 - (\cos\theta)/(\sqrt{(n^2 - \sin^2\theta)})]$

For $\theta = \theta_B$ = Brewster's angle, this reduces to

Eq. 44 $x = t[\sin\theta_B - (\cos\theta_B)/n]$

Displacement of secondary reflection from primary transmission, inclined plate

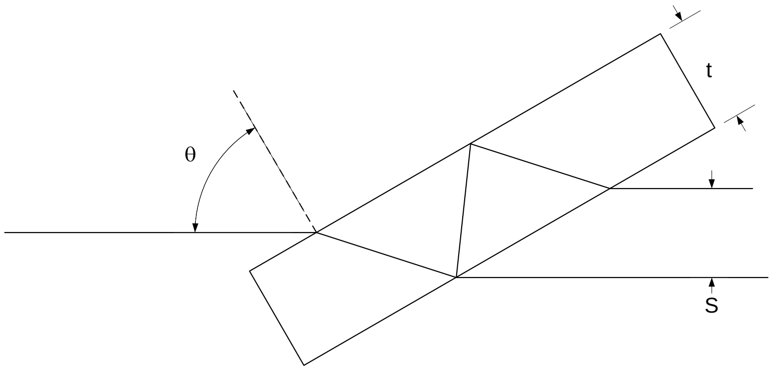

Figure 6: Beam separations after inclined plate

Eq. 45 $S = t \sin(2\theta)/[\sqrt{(n^2 - \sin^2\theta)}]$

Lenses

Lensmaker's formula

Thin lens versions

Eq. 46 $1/f = (n-1)(1/R_1 - 1/R_2)$

Eq. 47 $1/f = (n-1)/R$

Where f = focal length
R_1 = radius of 1st surface
R_2 = radius of 2nd surface
R = radius of curved surface if other surface is plano. $(1/\infty = 0)$
Radii are positive if their centers of curvature come *after* the surface.

To find the focal length of a plano-convex or plano-concave lens,

Eq. 48 $R = (n-1)f$

Thick lens version

Eq. 49 $1/f = (n-1)(1/R_1 - 1/R_2) + (n-1)^2/n * t_c/R_1R_2$

Where t_c = center thickness. Thickness does not factor into the focal length of plano-convex or plano-concave lenses.

Centering

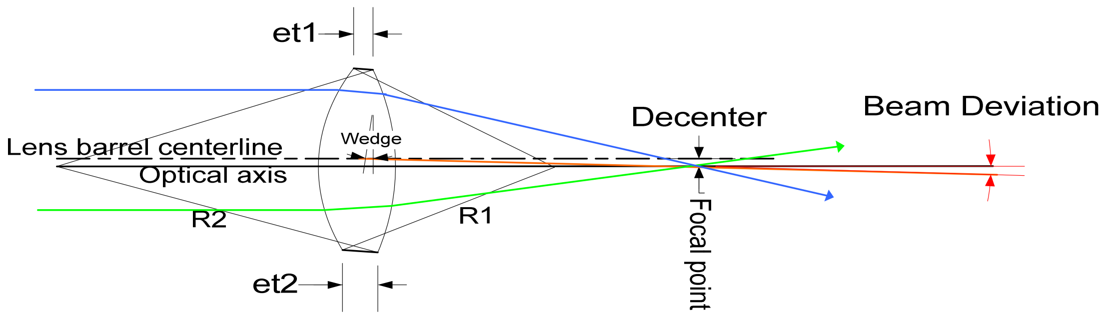

Figure 7: Decentered lens

f = focal length, Φ = diameter

To convert→ To ↓	Wedge angle **W**	ETV	Decentration **Δ**	Beam Deviation Angle **δ**
Wedge Angle **W**		$W = \arctan(ETV/\Phi)$	$W = \arctan(\Delta/R)$	$W = \delta/(n-1)$
Edge Thickness Variation ETV	$ETV = \Phi\tan(W)$		$ETV = \Phi*\Delta/R$	$ETV = \Phi\tan(\delta)/(n-1)$
Decentration **Δ**	$\Delta = \tan(W)*R$	$\Delta = R*ETV/\Phi$		$\Delta=f*\tan(\delta)$ Or $\Delta = \tan(\delta)/(n-1)$
Beam Deviation Angle δ	$\delta = (n-1)W$	$\delta = (n-1)*\arctan(ETV/\Phi)$	$\delta = \arctan(\Delta/f)$ Or $\delta = (n-1)\Delta/R$	

Table 1: Plano-spherical lenses

f = focal length, Φ = diameter, L = distance between centers of curvature (R1 – t_c - R2)

To convert→ To ↓	Wedge angle **W**	ETV	Decentration **Δ**	Beam Deviation Angle **δ**
Wedge Angle **W**		W = atan(ETV/Φ)	W = atan{Δ/[(n-1)*f] Or W = atan{Δ* [R2-R1+(n-1)*t_c/n] /R1*R2}	W = δ/(n-1)
Edge Thickness Variation ETV	ETV = Φtan(W)		ETV = ΔΦ/[(n-1)f]	ETV = Φtan(δ)/(n-1)
Image Decentration **Δ**	Δ = (n-1)tan(W)* f or Δ = R1*R2*tan(W) /[R2-R1+ (n-1)*t_c/n]	Δ = (n-1)* ETV*f/Φ		Δ=f*tan(δ)
Lateral displacement of lens when chucked on R2 to restore centration	Δ_E = R1R2*tan(W)/L	Δ_E = R_1R_2* ETV/(ΦL)	Δ_E = ΔR1R2 /[f(n-1)L]	Δ_E = R1R2*tan(δ) /[(n-1)L]
Lateral displacement of R1's center of curvature when chucked on R2	Δ_{R1} = R1tan(W)	Δ_{R1} = ETV*R1/Φ	Δ_{R1} = R1*Δ/[(n-1)f]	Δ_{R1} = R1tan(δ) /(n-1)
Beam Deviation Angle **δ**	δ = (n-1)W	δ = (n-1)* atan(ETV/Φ)	δ = atan(Δ/f)	

Table 2: Bi-curve lenses

Conversions

Temperature

Eq. 50 °C = (5/9)*(°F-32)
Eq. 51 °F = °C*9/5 + 32
Eq. 52 K = °C + 273.15

Where °C = temperature expressed in degrees Celsius (centigrade)
°F = temperature expressed in degrees Fahrenheit
K = absolute thermodynamic temperature in Kelvins[12]

Size of degrees

Difference in temperature of 1.8 °F = *difference* of 1 °C = *difference of* 1 K

A few important temperatures:

- - 40 °C = - 40 °F.
- Boiling point of water at sea level = 212 °F = 100 °C.

- Freezing point of water at sea level = 32 °F = 0 °C.
- Absolute zero (as cold as possible) = 0 K = -273.15 °C = -459.67 °F.

Length

1 inch = 25.4 mm

Mass

1 oz ≅ 28.349 g
1 lb ≅ 453.592 g
1 kg ≅ 2.2046 lb.

Angle

1 circle = 360°
1 circle = 2 π radians ≅ 6.2832 radians
1 radian ≅ 57.2958°
1 mrad ≅ 3.4377 minutes
1 μrad ≅ 0.2063 seconds
1 second ≅ 4.848 μrad
Brewster's angle = $\tan^{-1}(n)$

Geometry

Area

Square

Eq. 53 $A = s^2$

Where s = length of a side

Rectangle

Eq. 54 A = L * W

Where L = length and W = width

Parallelogram

Eq. 55 A = B * H

Where B = base and H = height perpendicular to base

Trapezoid

Eq. 56 $A = ½ (B_1 + B_2) * H$

Where B_1 and B_2 are the lengths of the two parallel edges, and H is the distance between the two parallel edges.

Triangle

Eq. 57 $A = \frac{1}{2} B * H$

Where B = base and H = height perpendicular to base. Any side can be taken as B and the distance from the opposite vertex along a perpendicular to B is then called H.

Circle

Eq. 58 $A = \pi r^2$

Where r = half the diameter of the circle.

Volume

Cube

Eq. 59 $V = s^3$

Where s = length of any side.

Brick[13]

Eq. 60 $V = L * W * H$

Where L = length, W = width, and H = height.

Cylinder

Eq. 61 $V = \pi r^2 h$

Where r = half the diameter of the cylinder, and H is the length measured along *the central axis.*

Note: The ends don't need to be perpendicular to the axis, or even parallel to each other.

Sphere

Eq. 62 $V = 4/3 \pi r^3$

Summary

Use this module as a reference in the shop. Don't trust little slips of paper that you see taped to the walls. Check back for updated versions.

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[1] There's another common form of this equation that looks a bit simpler, $h = R - \sqrt{(R^2 - y^2)}$. Beware of this version: When entering negative radii it will give a *very* wrong answer, and rounding errors in calculators reduce its accuracy for long radii. The form given in the main text can take negative radii and gives more accurate answers for long radii.

[2] I haven't seen this published. My derivation, April 1987.

[3] I haven't seen this published. My derivation, February 1984.

[4] I haven't seen this published. My derivation, June 1973.
[5] After much trigonometry, I discovered that this simple formula is correct within 2% in all cases, regardless whether the block starts with 1, 3, or 4 in the center.
[6] I haven't seen this published. Thanks to Teddi von der Ahe for this nice derivation, April 1990.
[7] "Whenever Two Beams Interfere, One Fringe Equals One Wave (in the Plane of Interference) – Always," Ray Williamson, SPIE Proceedings vol. 1527, 1991.
[8] Ibid., and private communication, David Kohler.
[9] All equations due to Norm Brown, Optical Fabrication and Testing Workshop, 3/15/1975.
[10] Sparks and Cottis, Pressure-induced optical distortion in laser windows, J. Appl. Phys, v. 44 #2, February 1973; and Roark and Young, Formulas for Stress and Strain, McGraw-Hill
[11] Poisson's ratio appears in several terms of these equations. Poisson's ratio *must* lie between 0 and 0.5, and almost all materials are between 0.09 and 0.4 with the great majority around 0.25. I've inserted 0.25 into the messier equations to achieve these approximations. The differences turn out to be small.
For the hardy, here's the full expression for a simply supported and clamped window:
$h_{ss} = 12Py^4[(5 + \nu)*(1 - \nu^2)/(1 + \nu)]/(64Et^3)$
$h_c = 12Py^4(1-\nu^2)/(64Et^3)$
Where ν = Poisson's ratio.
See for yourself how little difference it makes.
[12] Note that Kelvin is a basic unit, like grams and volts. The Kelvin scale starts at absolute zero, and the number of Kelvins above absolute zero is the thermodynamic temperature: 1000 K is twice as hot as 500 K. By contrast, the Celsius and Fahrenheit scales start at arbitrary points and so they're more like addresses: 1000 Elm Street is not "twice" 500 Elm Street in any respect, and 2 °C is definitely not twice as hot as 1 °C, either in absolute terms or our own perception. Therefore the Celsius and Fahrenheit scales are expressed in degrees and Kelvin is not.
The *difference in temperature* expressed by a difference of 1°C is equal to a difference of 1 K, and is also equal to a *difference in temperature* of 1.8 °F.
[13] OK, officially it's called a parallelepiped.